Propositional logic.

Syntax

Definition A *propositional language* L consists of of two parts:

(a) the logical connectives: $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$

(b) a non-empty set of symbols called *propositional variables*.

Since the logical connectives are common to all propositional languages, we shall usually just write $L = \{p_1, p_2, \ldots\}$, where p_i are the propositional variables of L.

Definition Suppose L is a propositional language. The set of *well-formed formulae* or just *formulae* of L, denoted by Form(L), is the smallest set such that

(a) every $P \in L$ is a wff,

(b) if A is a wff, then so is $\neg A$,

(c) if A and B are wff, then so are

$$(A \vee B) \ , \ (A \vee B) \ , \ (A \rightarrow B) \text{ and } (A \leftrightarrow B).$$

The rôle of propositional logic is to be able to formalise sentences and arguments in natural language and mathematics that have a particularly simple form.

We will use propositional variables to denote atomic propositions and use logical connectives to construct more complex statements.

<u>Interpretation of logical symbols</u>:

$$\neg A = \text{not } A$$

$$(A \vee B) = A \text{ or } B$$

$$(A \wedge B) = A \text{ and } B$$

$$(A \rightarrow B) = \text{if } A \text{, then } B$$

$$(A \leftrightarrow B) = A \text{ if and only if } B.$$

Example: Formalise the argument:

"If Obama is president, then Obama will reform health care. Obama will reform health care. Therefore, Obama is president."

We let $L = \{P, Q\}$ and interpret P and Q as:

P : Obama is president

Q : Obama will reform health care.

So we formalise this as:

If P, then Q. Q. Therefore P.

i.e.,

$$\frac{\begin{array}{c}(P \to Q) \\ Q\end{array}}{P}$$

Here the horizontal line denotes that a conclusion is drawn.

<u>Example</u> If Jones did not meet Smith last night, then either Jones was the murderer or Jones is lying. If Smith was not the murderer, then Jones did not meet Smith last night and the murder took place after midnight. If the murder took place after midnight, then either Smith was the murderer or Jones is lying. Hence, Smith was the murderer.

M = Jones met Smith

L = Jones is lying

J = Jones is murderer

S = Smith is murderer

A = Murder took place after midnight.

So formalised:

$$(\neg M \rightarrow (J \vee L))$$
$$(\neg S \rightarrow (\neg M \wedge A))$$
$$\underline{(A \rightarrow (S \vee L))}$$
$$S$$

Induction on the height of formulas

Given a propositional language L, we can give an explicit construction of the set $\mathrm{Form}(L)$ as follows.

We let $\mathcal{F}_0 = L$ = sets of prop. var.
and inductively

$$\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{\neg A, (A \vee B), (A \wedge B), (A \rightarrow B), (A \leftrightarrow B) : A, B \in \mathcal{F}_n\}.$$

Proposition $\mathrm{Form}(L) = \bigcup_n \mathcal{F}_n$.

The above definition allows us to define the *height* of a wff as follows:

$$h(A) = \min n \text{ s.t. } A \in \mathcal{F}_n .$$

This allows us to prove results about formulas by induction on the height.

Namely, suppose $\mathcal{S}$ is a set of wff s.t.

(a) $P \in \mathcal{S}$ for each prop. var. P

(b) if $A, B \in \mathcal{S}$, then also

$\neg A, (A \vee B), (A \wedge B), (A \rightarrow B), (A \leftrightarrow B) \in \mathcal{S}$

Then any wff belongs to $\mathcal{S}$

For otherwise, we choose a wff A of minimal height not belonging to $\mathcal{S}$.

By (a), we have $h(A) = n+1$ for $n \geq 0$.

So either $A = \neg B$ for some B & $h(B) = n$,

or $A = (B \vee C)$ etc. for B, C of height $\leq n$.
In any case, $B, C \in S$, whereby by (b) we also have $A \in S$, which is a contradiction.

Semantics

Suppose L is a propositional language. A valuation of L is a function v from the variables of L to the set $\{T, F\}$:

$$v : L \to \{T, F\}.$$

Given a valuation v of L, we can extend v to a function from Form(L) to $\{T, F\}$, also denoted by v, as follows:

(a) $$v(\neg A) = \begin{cases} F & \text{if } v(A) = T \\ T & \text{if } v(A) = F \end{cases}$$

(b) $$v((A \vee B)) = \begin{cases} F & \text{if } v(A) = v(B) = F \\ T & \text{otherwise.} \end{cases}$$

(c) $$v((A \wedge B)) = \begin{cases} T & \text{if } v(A) = v(B) = T \\ F & \text{otherwise} \end{cases}$$

(d) $$v((A \rightarrow B)) = \begin{cases} F & \text{if } v(A) = T \text{ and } v(B) = F \\ T & \text{otherwise} \end{cases}$$

(e) $$v((A \leftrightarrow B)) = \begin{cases} T & \text{if } v(A) = v(B) \\ F & \text{if } v(A) \neq v(B) \end{cases}$$

<u>Definition</u> Let $v : \mathcal{L} \to \{T, F\}$ be a valuation of a propositional language $\mathcal{L}$ and let $A \in \mathrm{Form}(\mathcal{L})$, i.e., A is a wff of $\mathcal{L}$.

We say that A is *true* under v if $v(A) = T$ and *false* otherwise.

<u>Example</u> Let $L = \{P, Q, R, S\}$ and

$$A = \Big((P \leftrightarrow Q) \longrightarrow (\neg R \leftrightarrow (S \vee P)) \Big)$$

Let $v : L \to \{T, F\}$ be given by

$$v(P) = T, \quad v(Q) = T, \quad v(R) = F, \quad v(S) = F$$

Then $v(\neg R) = T$, $v(S \vee P) = F$, $v((P \leftrightarrow Q)) = T$

$$v\big((\neg R \leftrightarrow (S \vee P))\big) = F, \quad v(A) = F$$

So A is false under v.

Truth tables

An efficient way of organising all valuations of a finite language L is via <u>truth tables</u>.

Namely, suppose $L = \{P, Q, R\}$ and

$A = ((P \vee Q) \rightarrow (R \vee (R \rightarrow Q)))$ is wff.

We draw the truth table for A as follows:

P	Q	R	$(R \to Q)$	$(R \vee (R \to Q))$	$(P \vee Q)$	A
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	F	T	T	T
T	F	F	T	T	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	F	T
F	F	F	T	T	F	T

Note that each row in the truth table corresponds to a unique truth valuation of L and, vice versa, each valuation corresponds to a row in the truth table.

<u>Definition</u> Let L be a propositional language and $\mathcal{A}$ a set of wff of L.

We say that $\mathcal{A}$ is <u>satisfied</u> by a valuation $v : L \to \{T, F\}$ of L if $v(A) = T$ for any $A \in \mathcal{A}$.
In this case, v is a <u>model</u> of $\mathcal{A}$.
$\mathcal{A}$ is <u>satisfiable</u> if it is satisfied by some valuation of L.

Remark If $\mathcal{A} \subseteq \mathcal{B}$ and v satisfies $\mathcal{B}$, then v also satisfies $\mathcal{A}$. So if $\mathcal{B}$ is satisfiable, then so is $\mathcal{A}$.

Note that $\mathcal{A}$ is satisfiable if and only if there is a row in the truth table that gives value T to all formulas of $\mathcal{A}$.

Definition

A wff A of a propositional language $\mathcal{L}$ is a tautology if $v(A) = T$ for every valuation v of $\mathcal{L}$.

Example The formulas

$$A = ((P \vee Q) \longrightarrow (R \vee (R \rightarrow Q)))$$

and

$$B = (R \vee (R \rightarrow Q))$$

are both tautologies of the language $\mathcal{L}$. This is seen from the preceding truth table.

Definition Let L be a propositional language, $\mathcal{B}$ a set of wff and A, B wff.

We say that A is a *tautological consequence* of $\mathcal{B}$, written $\mathcal{B} \models A$, if $v(A) = T$ whenever v is valuation satisfying $\mathcal{B}$.

We say that A and B are *tautologically equivalent* if $v(A) = v(B)$ for any valuation v. Denote this by $A \equiv B$.

Lemma Let P, Q, R be variables of a language L. Then

(a) $((P \wedge Q) \wedge R)$ and $(P \wedge (Q \wedge R))$ are tautologically equivalent.

(b) $((P \vee Q) \vee R)$ and $((P \vee Q) \vee R)$ are tautologically equivalent.

Pf We check the truth tables:

P	Q	R	$((P \wedge Q) \wedge R)$	$(P \wedge (Q \wedge R))$
T	T	T	T	T
T	T	F	F	F
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F	F	F

And similarly for $((P \vee Q) \vee R)$ and $(P \vee (Q \vee R))$. $\square$

It follows by induction that all distributions of parentheses in

$$P_1 \wedge P_2 \wedge \cdots \wedge P_n$$

lead to tautologically equivalent formulas.

We shall therefore allow ourselves to write conjunctions and disjunctions w/o parentheses.

Substitution in a formula

Theorem Suppose $A, B_1, \dots, B_n$ are formulas and $P_1, \dots, P_n$ propositional variables of a language L. Let A^* be obtained from A by simultaneously substituting every occurrence of P_i in A by B_i. Then A^* is a formula of L.

Moreover, if $v : L \to \{T, F\}$ is any valuation of L and $w : L \to \{T, F\}$ is defined by

$$w(Q) = \begin{cases} v(Q) & \text{if } Q \neq P_1, P_2, \dots, P_n \\ v(B_i) & \text{if } Q = P_i \end{cases},$$

then $v(A^*) = w(A)$.

Proof That A^* is a formula of L is proved by induction on the height of A and is left as a homework exercise.

Instead, we shall prove the moreover part by induction on $h(A)$.

$h(A) = 0$. In this case, A is a propositional variable Q. If $Q = P_i$ for some i, then $A^* = B_i$ and so

$$v(A^*) = v(B_i) = w(P_i) = w(A).$$

On the other hand, if $Q \neq P_i$ for all i, then $A^* = Q$ and thus

$$v(A^*) = v(Q) = w(Q) = w(A).$$

$h(A) = k+1$: Suppose the moreover part is true for all formulas A of height $\le k$. Assume $h(A) = k+1$.

Case 1: $A = \neg C$, where $h(C) = k$. Then $A^* = \neg C^*$ and by the induction hypothesis, $v(C^*) = w(C)$, so

$$v(A^*) = v(\neg C^*) = \begin{cases} T & \text{if } v(C^*) = F \\ F & \text{if } v(C^*) = T \end{cases}$$

$$= \begin{cases} T & \text{if } v(C) = F \\ F & \text{if } v(C) = T \end{cases} = \begin{cases} T & \text{if } v(A) = T \\ F & \text{if } v(A) = F \end{cases}$$

$= v(A)$.

<u>Case 2</u> $A = (C \vee D)$, where $h(C), h(D) \leq k$.

Now, $A^* = (C^* \vee D^*)$ and so

$$v(A^*) = \begin{cases} F & \text{if } v(C^*) = v(D^*) = F \\ T & \text{otherwise.} \end{cases}$$

Again by the induction hypothesis, $v(C^*) = w(C)$, $v(D^*) = w(D)$, so

$$v(A^*) = \begin{cases} F & \text{if } w(C) = w(D) = F \\ T & \text{otherwise.} \end{cases}$$

Since $A = (C \vee D)$, we see that

$v(A^*) = w(A)$.

The cases $A = (C \wedge D)$, $(C \rightarrow D)$, $(C \leftrightarrow D)$ are similar.

$\square$

Corollary Suppose A^* is obtained from A as before. Then if A is a tautology, so is A^*

Proof Suppose v is any valuation of A^* and define w as in the theorem. Then

$$v(A^*) = w(A) = T$$

since A is a tautology. So A^* is a tautology too. $\square$

Corollary Suppose A^* and B^* are obtained from simultaneous substitution of $B_1, \ldots, B_n$ for propositional variables $P_1, \ldots, P_n$ in formulas A, B. Then if

$$A \equiv B$$

also

$$A^* \equiv B^*$$

Examples (a) Since $((P \vee Q) \vee R) \equiv (P \vee (Q \vee R))$

for all propositional variables P, Q, R,

we also have

$$((A \vee B) \vee C) \equiv (A \vee (B \vee C))$$

for any formulas A, B, C.

(b) Using truth tables one sees that

$$(P \rightarrow (Q \rightarrow P))$$

is a tautology.

So also

$$((\neg R \leftrightarrow (P \vee S)) \rightarrow ((\neg R \vee S) \rightarrow (\neg R \leftrightarrow (P \vee S))))$$

is a tautology.

Truth Functions

Note that if $L = \{P_1, P_2, \ldots, P_n\}$ then there are exactly 2^n valuations of L.

Also if $Val(L)$ denotes the set of all valuations of L and A is any L-formula, then A defines a function

$$\varphi_A : Val(L) \longrightarrow \{T, F\}$$

given by $\varphi_A(v) := v(A)$.

<u>Definition</u> A <u>truth function</u> on L is a function

$$\varphi : Val(L) \longrightarrow \{T, F\},$$

i.e., φ assigns a value T or F to any valuation v.

For example, φ_A above is a truth function.

<u>Remark</u> Since $|Val(L)| = 2^{|L|}$ for any language L, we see that there are $2^{2^{|L|}}$ truth functions on L.

<u>Example</u> $L = \{P\}$. In this case, there are two valuations v_1, v_2 and four truth functions $\varphi_1, \varphi_2, \varphi_3, \varphi_4$.

	v_1	v_2	Example of formula A s.t.
P	T	F	$\varphi_A = \varphi_j$
$\varphi_1(v_i)$	T	T	$(P \vee \neg P)$
$\varphi_2(v_i)$	T	F	P
$\varphi_3(v_i)$	F	T	$\neg P$
$\varphi_4(v_i)$	F	F	$(P \wedge \neg P)$

<u>Example</u> $L = \{P, Q\}$.

We now have four valuations v_1, v_2, v_3, v_4 and 16 truth functions, $\varphi_1, \ldots, \varphi_{16}$

	v_1	v_2	v_3	v_4	Example of a formula A s.t. $\varphi_A = \varphi_j$
P	T	T	F	F	
Q	T	F	T	F	
$\varphi_1(v_i)$	T	T	T	T	$(P \vee \neg P)$
$\varphi_2(v_i)$	T	T	T	F	$(P \vee Q)$
$\varphi_3(v_i)$	T	T	F	T	$(P \rightarrow Q)$
$\varphi_4(v_i)$	F	F	F	T	$(\neg P \wedge \neg Q)$ (neither nor)
$\varphi_5(v_i)$	F	T	T	T	$\neg(P \wedge Q)$ (not both)
$\vdots$					

We leave it as a homework exercise to fill out the remainder of the table.

Conjunctive and disjunctive normal forms.

Let L be a propositional language.

A literal of L is either a propositional variable or the negation of a propositional variable.

A formula A is said to be in conjunctive normal form if there are literals B_{ij} s.t.

$$(*)\left\{\begin{array}{l} A = (B_{11} \vee B_{12} \vee \dots \vee B_{1n_1}) \wedge (B_{21} \vee B_{22} \vee \dots \vee B_{2n_2}) \\ \qquad \wedge \dots \wedge (B_{m1} \vee B_{m2} \vee \dots \vee B_{mn_m}) \end{array}\right.$$

Observation Suppose A is in conjunctive normal form and $v : L \to \{T, F\}$ is a valuation. Then

$v(A) = T \iff$ for every $i \le m$ there is $j \le n_i$ st. $v(B_{ij}) = T$.

Similarly, a formula A is in <u>disjunctive normal form</u> if there are literals B_{ij} s.t.

$$(**)\ \left\{ \begin{array}{l} A = (B_{11} \wedge B_{12} \wedge \dots \wedge B_{1n_1}) \vee (B_{21} \wedge \dots \wedge B_{2n_2}) \\ \qquad \vee \dots \vee (B_{m1} \wedge \dots \wedge B_{mn_m}) . \end{array} \right.$$

<u>Theorem</u> Let $L = \{P_1, \dots, P_n\}$ be a finite language and $\varphi : Val(L) \longrightarrow \{T, F\}$ be a truth function. Then there is a formula A in disjunctive normal form st. $\varphi = \varphi_A$

<u>Proof</u> If $\varphi \equiv F$, then we can let $A = (P_1 \wedge \neg P_1)$, whence $\varphi_A \equiv F$ and thus $\varphi = \varphi_A$.

So suppose instead that $\varphi(w) = T$ for some w. Now list all valuations such that $\varphi(v) = T$ as $v_1, v_2, \dots, v_m$.

For $i \le m$ and $j \le n$ set $B_{ij} = \begin{cases} P_j & \text{if } v_i(P_j) = T \\ \neg P_j & \text{if } v_i(P_j) = F. \end{cases}$

Then we see that $A_i = (B_{i1} \wedge B_{i2} \wedge \dots \wedge B_{in})$

is satisfied by a valuation v if and only if

$$v(B_{i1}) = v(B_{i2}) = \ldots = v(B_{in}) = T, \text{ i.e.,}$$

if $v(P_j) = v_i(P_j)$ for all j.

So $v(A_i) = T \iff v = v_i$.

Hence for $A = A_1 \vee \ldots \vee A_m$,

$$v(A) = T \iff \exists i \le m \quad v(A_i) = T$$

$$\iff \exists i \le m \quad v = v_i.$$

Thus $f_A(v) = f(v)$ for all $v \in Val(L)$ and thus $f_A = f$. $\square$

__Remark__ Note that two formulas A and B are tautologically equivalent, $A \equiv B$, if and only if $f_A = f_B$.

__Corollary__ Any formula is tautologically equivalent to one in disjunctive normal form.

<u>Proof</u> If B is any formula, we find a formula A in disjunctive normal form such that $f_A = f_B$. Then for any valuation $v: L \to \{T, F\}$ we have

$$v(A) = f_A(v) = f_B(v) = v(B)$$

and so $A \equiv B$. $\square$

<u>Corollary</u> Any formula is tautologically equivalent to one in conjunctive normal form.

<u>Proof</u> Let A be any formula of $L = \{P_1, \dots, P_n\}$ and find literals B_{ij} s.t.

$$\neg A \equiv (B_{11} \wedge \dots \wedge B_{1k_1}) \vee \dots \vee (B_{m1} \wedge \dots \wedge B_{mk_m}).$$

One easily checks that then

$$A \equiv \neg\neg A \equiv \neg\left((B_{11} \wedge \dots \wedge B_{1k_1}) \vee \dots \vee (B_{m1} \wedge \dots \wedge B_{mk_m})\right)$$

$$\equiv \neg(B_{11} \wedge \dots \wedge B_{1k_1}) \wedge \dots \wedge \neg(B_{m1} \wedge \dots \wedge B_{mk_m})$$

$$\equiv (\neg B_{11} \vee \dots \vee \neg B_{1k_1}) \wedge \dots \wedge (\neg B_{m1} \vee \dots \vee \neg B_{mk_m}).$$

Now, the B_{ij} are literals, i.e. $B_{ij} \equiv P$ or $B_{ij} \equiv \neg P$ for some prop. var. P, so

$$\neg B_{ij} \equiv \neg P \quad \text{or} \quad \neg B_{ij} \equiv P, \text{ resp.}$$

Replacing the $\neg B_{ij}$ with literals as above, we see that A is tautologically equivalent to one in conjunctive normal form. $\Box$

<u>Remark</u> We have shown that any formula is equivalent to a formula in disjunctive or conjunctive normal form and hence to a formula only using the connectives $\neg$, $\wedge$ and $\vee$.

However, using that $(A \vee B) \equiv \neg(\neg A \wedge \neg B)$, we see that we can eliminate $\vee$ from the disjunctive normal form and hence get a tautologically equivalent formula only containing the connectives $\wedge$ and $\neg$.